Syllabus: Superposition of two perpendicular oscillations for phase difference $0, \pi/2, \pi$, Graphical and analytical methods, Lissajous figures with equal and unequal frequencies

TWO MUTUALLY PERPENDICULAR HARMONIC OSCILLATIONS HAVING SAME FREQUENCIES

Let’s assume that two independent forces acting on a particle in such a manner that the first alone produces a simple harmonic motion in the x-direction given by

$$x = A_1 \sin \omega t$$

and the second would produce a simple harmonic motion in the y-direction given by

$$y = A_2 \sin(\omega t + \delta)$$

Thus, we are actually considering the superposition of two mutually perpendicular SHMs which have equal frequencies. The amplitudes are different and their phases differ by $\delta$. The resultant motion of the particle is a combination of the two SHMs.

The position of the particle at any time $t$ is given by $(x, y)$ where $x$ and $y$ are given by the above equations. The resultant motion is, thus, two-dimensional and the path of the particle is, in general, an ellipse. The equation of the path traced by the particle is obtained by eliminating $t$ from above equations.

Now,

$$\sin \omega t = \frac{x}{A_1} \text{ and } \cos \omega t = \sqrt{1 - \frac{x^2}{A_1^2}}$$

Putting these values in equation for $y$,

$$y = A_2 (\sin \omega t \cos \delta + \cos \omega t \sin \delta) = A_2 \left( \frac{x}{A_1} \cos \delta + \sqrt{1 - \frac{x^2}{A_1^2}} \sin \delta \right)$$

Or,

$$\frac{y}{A_2} = \left( \frac{x}{A_1} \cos \delta + \sqrt{1 - \frac{x^2}{A_1^2}} \sin \delta \right)$$

Or,

$$\frac{y}{A_2} - \frac{x}{A_1} \cos \delta = \sqrt{1 - \frac{x^2}{A_1^2}} \sin \delta$$

Or,

$$\left( \frac{y}{A_2} - \frac{x}{A_1} \cos \delta \right)^2 = \left( \sqrt{1 - \frac{x^2}{A_1^2}} \sin \delta \right)^2$$
Or, \[ \frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} \cos^2 \delta - \frac{2xy}{A_1A_2} \cos \delta = \left(1 - \frac{x^2}{A_1^2}\right) \sin^2 \delta \]

Or, \[ \frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} \cos^2 \delta + \frac{x^2}{A_1^2} \sin^2 \delta - \frac{2xy}{A_1A_2} \cos \delta = \sin^2 \delta \]

Or, \[ \frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} - \frac{2xy}{A_1A_2} \cos \delta = \sin^2 \delta \]

\[ \text{-------------------------(1)} \]

This is an equation of ellipse. Thus, we may conclude that the resultant motion of the particle is along an elliptical path.

Here \( x \) remains between \(-A_1\) and \( A_1\) and that of \( y \) remains between \(-A_2\) and \( A_2\). Thus, the particle always remains inside the rectangle defined by \( x = \pm A_1 \) and \( y = \pm A_2 \).

The ellipse can be drawn as,

\[ \text{Special Cases} \]

The two component SHMs are in phase, \( \delta = 0 \)

The two component SHMs are out of phase, \( \delta = \pi \)

The phase difference between the two component SHMs, \( \delta = \pi/2 \)

Let us now obtain the resultant motion of the particle under these special cases.

\[ \text{When the two superposing SHMs are in phase, } \delta = 0 \Rightarrow \cos \delta = 1, \quad \sin \delta = 0 \]

\[ \text{Eq}(1) : \quad \frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} - \frac{2xy}{A_1A_2} = 0 \quad \Rightarrow \left(\frac{y}{A_2} - \frac{x}{A_1}\right)^2 = 0 \quad \Rightarrow \frac{y}{A_2} - \frac{x}{A_1} = 0 \]
This is an equation of a straight line passing through the origin and have a slope of $\tan^{-1}\left(\frac{A_2}{A_1}\right)$.

The path of the particle is:

![Diagram of a straight line passing through the origin with a slope](image)

For $\delta=0$, $x = A_1 \sin \omega t$, $y = A_2 \sin \omega t \Rightarrow r = \sqrt{A_1^2 + A_2^2} \sin \omega$

Thus, we can see that the resultant motion is also SHM with the same frequency and phase as the component motions. However, the amplitude of the resultant SHM is $\sqrt{A_1^2 + A_2^2}$.

When the two superposing SHMs are in phase, $\delta = \pi \Rightarrow \cos \delta = -1$, $\sin \delta = 0$

Eq(1): $\frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} + \frac{2xy}{A_1A_2} = 0 \Rightarrow \left(\frac{y}{A_2} + \frac{x}{A_1}\right)^2 = 0 \Rightarrow \frac{y}{A_2} + \frac{x}{A_1} = 0$

This is an equation of a straight line passing through the origin and have a slope of $\tan^{-1}\left(-\frac{A_2}{A_1}\right)$

The path of the particle is:

![Diagram of a straight line passing through the origin with a slope](image)

When the phase difference between the two component SHMs is $\delta = \pi/2$.

$\Rightarrow \cos \delta = 0$, $\sin \delta = 1$
Eq(1): \[ \frac{y^2}{A_2^2} + \frac{x^2}{A_1^2} = 1 \]

This is an equation of ellipse with its axes along the x-axis and the y-axis and with its center at the origin. The lengths of the major and the minor axes are $2A_1$ and $2A_2$, respectively. The particle will move in an ellipse.

In case the amplitudes of the two individual SHMs are equal, $A_1 = A_2 = A$, i.e. the major and the minor axes are equal, then the ellipse reduces to a circle.

\[ x^2 + y^2 = A^2 \]

Thus, the resultant motion of a particle due to superposition of two mutually perpendicular SHMs of equal amplitude and having a phase difference of $\pi/2$ is a circular motion. The circular motion may be clockwise or anticlockwise depending on which component leads the other.

**Example 1**: Show that the superposition of oscillations represented by $x = A\sin\omega t$, $y = -A\cos\omega t$ results in to circular motion traced in the anticlockwise sense.

**Solution**: Here phase difference $\delta = 0$, particle moves in path $x^2 + y^2 = A^2$, which is a circle of radius A.

At $t=0$, $x = 0$, $y = -A$

At $t=T/4$, $x = \frac{A\sin\omega 2\pi}{4\omega} = \frac{A\sin\pi}{2} = A$, $y = -A\cos\omega 2\pi/4\omega = -A\cos\pi/2 = 0$
i.e motion of the particle is anticlockwise.

**Example 2:** A particle is subjected to two simple harmonic oscillations, one along the x-axis and the other on a line making an angle of \(\pi/4\) with the x-axis. The two motions are given by 
\[x = A \sin \omega t\]  and  \[S = B \sin \omega t\]. Calculate the amplitude of the resultant motion.

Solution: The two individual SHMs, with equal frequencies, phase difference between them \(\delta = 0\). Now, let’s break the second SHM into its x-component and the y-component.

\[
\begin{align*}
\text{Components along X-axis:} & \quad \frac{S}{\sqrt{2}} + A \sin \omega t = \frac{B \sin \omega t}{\sqrt{2}} + A \sin \omega t = \left(A + B \frac{\sin \omega t}{\sqrt{2}}\right) \\
\text{Component along Y-axis:} & \quad \frac{S}{\sqrt{2}} = \frac{B \sin \omega t}{\sqrt{2}}
\end{align*}
\]

The amplitude of the resultant oscillation for mutually perpendicular oscillations with zero phase difference is given by
Exercise:

1. Show that the superposition of oscillations represented by $x = A \sin \omega t$ and $y = A \cos \omega t$ results in circular motion traced in the clockwise sense.

2. A particle is subjected to two simple harmonic motions in the perpendicular directions having equal amplitudes (0.01 m) and equal frequencies. If the two SHMs are out of phase, what is the nature of the path followed by the particle?

3. A particle is subjected to two simple harmonic motions in the perpendicular direction having equal amplitudes (0.01 m) and equal frequencies. If the two component SHMs are in phase, what is the nature of path followed by the particle?

4. A particle is subjected to two SHMs $y = \sin \omega t$ and $z = \sin \omega t$. What is the path that the particle follows?

5. A particle is subjected to two simple harmonic oscillations, one along the $x$-axis and the other on a line making an angle of $3\pi/4$ with the $x$-axis. The two motions are given by $x = A \sin \omega t$ and $S = B \sin \omega t$. Calculate the amplitude of the resultant motion.

Oscillations Having Different Frequencies (Lissajous Figures)

When the frequencies of the two perpendicular SHMs are not equal, the resulting motion becomes more complicated. Let us suppose that the displacements of the two mutually perpendicular oscillations are given by

$$x = A_1 \sin \omega_1 t$$

$$y = A_2 \sin (\omega_2 t + \delta)$$

The phase difference between them at any instant $t$, is given by

$$\Delta = (\omega_2 t + \delta) - \omega_1 t = (\omega_2 - \omega_1) t + \delta$$

Since the superimposed orthogonal oscillations are of different frequencies, one of them changes faster than the other and will gain in phase over the other. As a result, the resultant motion passes through different phases. The phase of resultant changes with time due to the change in the phase difference of superimposed oscillations. However, the motion is confined within a rectangle of sides $2A_1$ and $2A_2$.

The pattern of the resultant motion is described by a *Lissajous figure*, named after French mathematician. *Lissajous figures can be seen by using a cathode ray oscilloscope (CRO).*
Here, two rectangular oscillations are simultaneously imposed upon a beam of cathode ray by connecting two sources of electrical oscillations to horizontal plates XX and vertical plates YY of the oscilloscope. We then see the trace of the resultant effect in the form of an electron beam on the fluorescent screen. By adjusting the phases, amplitudes and the ratio of the frequencies of the applied voltage, we obtain various curves. Lissajous figures may be used to compare two nearly equal frequencies. If the frequencies of two component oscillations are not exactly equal, the Lissajous figure will change gradually.

Let us now look at few examples to illustrate the shape of the Lissajous figure for some special cases.

Suppose a particle is subjected to two mutually perpendicular simple harmonic oscillations,

\[ x = A_1 \cos \omega_1 t \]
\[ y = A_2 \cos (\omega_2 t + \delta) \]
If you try to trace the trajectory of the motion of the particle using analytical method, i.e., to find $y$ in terms of $x$, eliminating $t$, calculation becomes very cumbersome if the phase difference $\delta$ is nonzero. Below is a table showing lissajous figure for $\delta=0$, $\omega_1$ and $\omega_2$ varying in the ratio $1:1$ (both equal), $1:2$ ($\omega_2=2\omega_1$), $1:3$ ($\omega_2=3\omega_1$)………..

Here amplitudes are taken equal ($A_1 = A_2$)

**Example 3**: A particle is subjected to two mutually perpendicular simple harmonic oscillations,

\[
x = A_1 \cos\omega t
\]
\[
y = A_2 \cos(2\omega t + \delta)
\]
Here ratio of frequency is 1:2

Eliminating t from these two equations, we can determine the equation of the trajectory of the particle, i.e. we are looking for an expression for y in terms of x.

\[ \cos \omega t = \frac{x}{A_1} \]

Now, \[ y = A_2 \cos(2\omega t + \delta) = A_2[\cos 2\omega t \cos \delta - \sin 2\omega t \sin \delta] \]

\[ = A_2[(2 \cos^2 \omega t - 1) \cos \delta - 2 \sin \omega t \cos \omega t \sin \delta] \]

\[ = A_2 \left[ \left(2 \frac{x^2}{A_1^2} - 1\right) \cos \delta - 2 \frac{x}{A_1} \sqrt{1 - \frac{x^2}{A_1^2}} \sin \delta \right] \]

\[ \frac{y}{A_2} = \left(2 \frac{x^2}{A_1^2} - 1\right) \cos \delta - 2 \frac{x}{A_1} \sqrt{1 - \frac{x^2}{A_1^2}} \sin \delta \]

\[ \frac{y}{A_2} + \cos \delta - \left(2 \frac{x^2}{A_1^2}\right) \cos \delta = -2 \frac{x}{A_1} \sqrt{1 - \frac{x^2}{A_1^2}} \sin \delta \]

Squaring,

\[ \left(\frac{y}{A_2} + \cos \delta\right)^2 + \left(2 \frac{x^2}{A_1^2}\right)^2 \cos^2 \delta - 2 \left(\frac{y}{A_2} + \cos \delta\right) 2 \frac{x^2}{A_1^2} \cos \delta = \left(2 \frac{x}{A_1} \sqrt{1 - \frac{x^2}{A_1^2}} \sin \delta\right)^2 \]

Or, \[ \left(\frac{y}{A_2} + \cos \delta\right)^2 + 4 \left(\frac{x^2}{A_1^2}\right)^2 \cos^2 \delta - 4 \left(\frac{y}{A_2} + \cos \delta\right) \frac{x^2}{A_1^2} \cos \delta = 4 \left(\frac{x}{A_1}\right)^2 \left(1 - \frac{x^2}{A_1^2}\right) \sin^2 \delta \]

or, \[ \left(\frac{y}{A_2} + \cos \delta\right)^2 + 4 \frac{x^4}{A_1^4} \cos^2 \delta - 4 \left(\frac{y}{A_2} + \cos \delta\right) \frac{x^2}{A_1^2} \cos \delta = 4 \left(\frac{x}{A_1}\right)^2 \sin^2 \delta \]

Or, \[ \left(\frac{y}{A_2} + \cos \delta\right)^2 + 4 \left(\frac{x^4}{A_1^4}\right)(\cos^2 \delta + \sin^2 \delta) - 4 \left(\frac{y}{A_2} + \cos \delta\right) \frac{x^2}{A_1^2} \cos \delta = 4 \left(\frac{x}{A_1}\right)^2 \sin^2 \delta \]

Or, \[ \left(\frac{y}{A_2} + \cos \delta\right)^2 + 4 \left(\frac{x^4}{A_1^4}\right) - 4 \left(\frac{y}{A_2} + \cos \delta\right) \frac{x^2}{A_1^2} \cos \delta = 4 \left(\frac{x}{A_1}\right)^2 \sin^2 \delta \]

Or, \[ \left(\frac{y}{A_2} + \cos \delta\right)^2 + 4 \left(\frac{x^2}{A_1^2}\right) - \left(\frac{y}{A_2} + \cos \delta\right) \cos \delta - \sin^2 \delta = 0 \]
Or, \[ \left( \frac{y}{A_2} + \cos \delta \right)^2 + 4 \left( \frac{x^2}{A_1^2} \right) \left[ \frac{x^2}{A_1^2} - \frac{y}{A_2} \cos \delta - (\cos^2 \delta + \sin^2 \delta) \right] = 0 \]

Or, \[ \left( \frac{y}{A_2} + \cos \delta \right)^2 + 4 \left( \frac{x^2}{A_1^2} \right) \left( \frac{x^2}{A_1^2} - \frac{y}{A_2} \cos \delta - 1 \right) = 0 \]

The above equation is of fourth degree, which, in general, represents a closed curve having two loops. For a given value of \( \delta \), the curve corresponding to the above expression can be traced using the knowledge of coordinate geometry.

For \( \delta = 0, \cos \delta = 1 \), we get \[ \left( \frac{y}{A_2} + 1 \right)^2 + 4 \left( \frac{x^2}{A_1^2} \right) \left[ \frac{x^2}{A_1^2} - \frac{y}{A_2} - 1 \right] = 0 \]

Or, \[ \left( \frac{y}{A_2} + 1 \right)^2 + \left( \frac{2x^2}{A_1^2} \right)^2 - 2 \left( \frac{2x^2}{A_1^2} \right) \left( \frac{y}{A_2} + 1 \right) = 0 \]

Or, \[ \left( \frac{y}{A_2} + 1 - \frac{2x^2}{A_1^2} \right)^2 = 0 \]

Or, \[ \frac{y}{A_2} + 1 = \frac{2x^2}{A_1^2} \]

This represents two coincident parabolas with their vertices at \((0,-A_2)\).

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**Example 4**: A particle is subjected to two mutually perpendicular simple harmonic oscillations,

\[
x = 2 \cos t \\
y = \cos (t + 4)
\]

Trace the trajectory of the particle using graphical method.

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In other cases, the resultant analytical solution becomes very cumbersome if the phase difference \( \delta \) is non-zero. In such motion, one can be constructed quite conveniently by using graphical method.
Graphical method

Here, we set up a table of values to see what is happening. We give each point a "point number" so that it is easier to understand when we graph the curve.

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>π/4</th>
<th>π/2</th>
<th>3π/4</th>
<th>π</th>
<th>5π/4</th>
<th>3π/2</th>
<th>7π/4</th>
<th>2π</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>2</td>
<td>1.4</td>
<td>0</td>
<td>-1.4</td>
<td>-2</td>
<td>-1.4</td>
<td>0</td>
<td>1.4</td>
<td>2</td>
</tr>
<tr>
<td>y</td>
<td>-0.6</td>
<td>0.1</td>
<td>0.7</td>
<td>1</td>
<td>0.7</td>
<td>-0.1</td>
<td>-0.8</td>
<td>-1</td>
<td>-0.7</td>
</tr>
<tr>
<td>Pt.no.</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

From the above table, the resulting curve, with the numbered points included, is shown in the following figure. Point 1 is actually equivalent to Point 9.

Exercise

1. Construct the Lissajous figures for the following component oscillations. If you are using graphical method, you may have to take more than 9 points to get the complete graph in some cases.

   (a) \( x=2\sin t, y=\cos 2t \)
   
   (b) \( x=\sin t, y=\cos(t+\pi/4) \)
   
   (c) \( x=\sin \pi t, y=2\sin(\pi t+\pi/2) \)

2. Any periodic motion, regardless of its complexity, can be reduced to the
sum of a number of simple harmonic motions by the application of the superposition principle. Is the above statement true or false?

3. A body is executing simple harmonic motion, and its displacement at time $t$ is given by

$$x = 5 \sin 3\pi t$$

Plot the displacement, velocity, and acceleration for two complete periods.

4. A particle is simultaneously subjected to two simple harmonic motions in the same direction in accordance with the following equations:

$$y_1 = 8\sin 2\pi t$$ and $$y_2 = 4\sin 6\pi t$$

Show graphically the resultant path of the particle.

5. A particle is subjected to three simple harmonic oscillations, one along the x-axis, second along the y-axis and the third along the z-axis. The three motions are given by

$$x = A\sin\omega t, \quad y = B\sin\omega t, \quad z = C\sin\omega t.$$  

Calculate the amplitude of the resultant motion.

6. Choose the correct one.

A particle moves on the x-axis according to the equation $x = A + B\sin\omega t$

The motion is a SHM with amplitude

(a) $A$ (b) $B$ (c) $A+B$ (d) $\sqrt{A^2 + B^2}$

7. Choose the correct option.

More than one choice can be correct.

Which of the following will change the time period as they are taken to Mars?

(a) A simple pendulum (b) A compound pendulum (c) LC circuit (d) A torsional pendulum (e) A spring-mass system

8. A particle is under the influence of two simultaneous SHMs in mutually perpendicular directions given by $x = \cos\pi t$, $y = \cos\pi t/2$ determine the trajectory of the resulting motion of the particle.